Inducing Implication Relations

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ABSTRACT

This study deals with learning implication relations from certain kinds of knowledge (statistical studies, possibilistic information, or possible worlds) about truth valuations on a set of propositions. An induction principle is considered in each one of these cases. If only the possible valuations are known, then we check when two families of possible valuations produce the same implication. A conditional probability can be deduced when probabilistic information about possible valuations is added. In particular, by applying equiprobability when this information is unknown, each preorder can be extended to a "causal" network that contains it. Changing probabilistic for possibilistic information, a conditional necessity is suggested. These conditional measures extend the classical case if the respective information is forgotten. Finally, material implication is studied in multiple-valued logic.

KEYWORDS: ***Material implication, preorder, possibility, probability, fuzzy preorders***

1. INTRODUCTION

In recent years, some interesting research on induction inference has been developed in the framework of learning [1]. Induction is the process of inferring a general rule (implication relation) from a group of specific cases (examples) and thus, informally, it can be defined as the relation between the truth in each possible case and the implication relation. More specifically, let $\mathscr{F}$ be the set of facts (statements) being relevant to a certain problem according to expert opinion. In each case k in which the problem is considered, each fact $x \in \mathscr{F}$ will have a truth value $\tau_k(x) \in V$, where V stands for a complete lattice and τ_k is called the valuation of the case k. Now if K is the set of all possible cases, the TRUTH on $\mathscr{F}$ is

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defined as the set of all possible valuations:

$$\mathfrak{T} = \{\tau_k : \mathscr{F} \to V/k \in K\}.$$

Thus, the TRUTH on $\mathscr{F}$ is modeled as a class of V-fuzzy subsets of $\mathscr{F}$.

On the other hand, we may introduce a conditional relation on $\mathscr{F}$, the implication relation I, which is modeled as a V-fuzzy binary relation, i.e.,

$$I : \mathscr{F} \times \mathscr{F} \to V;$$

where $I(b/a) \in V$ represents the truth value for the proposition "b is a consequence of a". Some results of this paper can be found in a more general framework in [2], where the conditional is a mapping from a subset of facts to facts.

Now the induction can be defined as the process to translate some information about $\mathfrak{T}$ into information about I. This study analyzes the induction process for (1) a set of statements from a bivalued or multiple valued logic [3] or (2) a family of probabilistic pieces of information, and (3) a family of possibilistic pieces of information.

The logic induction principle will be dealt with in section 1. This associates an implication relation $I(\mathfrak{T})$ to each family of valuation $\mathfrak{T} = \{\tau_k : \mathscr{F} \to V/k \in K\}$. We will study when two families of valuations produce the same implication relation.

The standard conditional induced from statistical information is analyzed in section 2. We will use it in order to obtain a probabilistic extension for each preorder relation on $\mathscr{F}$. The translation of these developments to a possibilistic environment is presented in section 3.

2. MATERIAL IMPLICATION

The logical induction principle [4] will be used to obtain implication relations from the family of possible valuations of a set of propositions. Moreover, it will be shown that preorders (abstractions of implication relations) are in one to one correspondence with closure and co-closure systems by this principle. Finally the situation when two families of valuations produce the same implication relation is studied.

Let $\mathscr{F}$ be a non-empty set of propositions taking truth values in a complete lattice V. Two particular cases will be considered, the bivalued case -$V = \{0, 1\}$- and the multiple valued case -$V = [0, 1]$-. Let us remark that a valuation $\tau : \mathscr{F} \to V$ can be identified with the respective V-fuzzy set. In the bivalued case, τ is identified with the set of true propositions $\tau = \{a \in \mathscr{F} : \tau(a) = 1\}$.

2.1. Induction Principle

In the bivalued logic, the induction principle associates the material implication (a implies b if b is true whenever a is true) to the family of true propositions in each possible case.

DEFINITION 1. *Let $\mathscr{C}$ be a family of subsets of $\mathscr{F}$. We will denote $I(\mathscr{C})$ the binary relation on $\mathscr{F}$ established by*

$$aI(\mathscr{C})b \text{ if and only if } b \in s \text{ for each } s \in \mathscr{C} \text{ such that } a \in s.$$

$I(\mathscr{C})$ will be called the implication relation induced from $\mathscr{C}$.

Having fixed a t-norm $*$[13], *the induction principle can be extended to a multiple valued case through the pseudo-inverse of* $*$:

DEFINITION 2. [5] *Given a t-norm* $*$, *the pseudo-inverse of* $*$ *is the mapping* $\otimes : [0,1] \times [0,1] \to [0,1]$ *defined by*:

$$a \otimes b = \sup\{z \in [0,1] : a^*z \leq b\}, \quad \forall a, b \in [0,1].$$

DEFINITION 3. *Let $\mathscr{C}$ be a family of fuzzy subsets of $\mathscr{F}$. We will denote $I^*(\mathscr{C})$ the fuzzy relation on $\mathscr{F}$, $I^*(\mathscr{C}) : \mathscr{F} \times \mathscr{F} \to [0,1]$, defined by*

$$I^*(\mathscr{C})(a,b) = \inf\{s(a) \otimes s(b) : s \in \mathscr{C}\}.$$

$I^*(\mathscr{C})$ will be called the $*$-implication relation induced from $\mathscr{C}$.

Remark 1. Let us observe that $I^*(\mathscr{C})$ is a generalization of $I(\mathscr{C})$. Concretely, if $\mathscr{C}$ is composed of classical (non-fuzzy) subsets then $I^*(\mathscr{C}) = I(\mathscr{C})$ for any t-norm $*$.

The properties of $I^*(\mathscr{C})$ and $I(\mathscr{C})$ are summarized as follows.

PROPOSITION 1.

1. *$I(\mathscr{C}) = I(\mathscr{C}_d)_d$ where $\mathscr{C}_d = \{\mathscr{F} - s / s \in \mathscr{C}\})$.*
2. *If $*$ is lower semicontinuous, then $I^*(\mathscr{C})$ is a fuzzy preorder.*
3. *$I(\mathscr{C})$ is a preorder.*
4. *If $\mathscr{C}_1 \subseteq \mathscr{C}_2$ then $I^*(\mathscr{C}_2) \subseteq I^*(\mathscr{C}_1)$.*
5. *If $\mathscr{C}_1 \subseteq \mathscr{C}_2$ then $I(\mathscr{C}_2) \subseteq I(\mathscr{C}_1)$.*

Proof

1. $aI(\mathscr{C}_d)b$ if and only if $bI(\mathscr{C}_d)a$, if and only if $\forall s \in \mathscr{C}\ b \notin s \Rightarrow a \notin s$, if and only if $\forall s \in \mathscr{C}\ a \in s \Rightarrow b \in s$, if and only if $aI(\mathscr{C})b$.
2. see [5].
3. is a consequence of 2.
4. Let $\mathscr{C}_1 \subseteq \mathscr{C}_2$. Given $a, b \in \mathscr{F}$, $I^*(\mathscr{C}_2)(a,b) = \inf\{s(a) \otimes s(b) : s \in \mathscr{C}_2\} \leq \inf\{s(a) \otimes s(b) : s \in \mathscr{C}_1\} = I^*(\mathscr{C}_1)(a,b)$.
5. is a consequence of 4. ■

2.2. Possible Sets of True Facts Associated with an Implication Relation

As an inverse process we can define the set of valuations which are compatible with a partial implication relation R (if aRb then a implies b):

DEFINITION 4. [6] *Given a binary relation R on $\mathscr{F}$, a subset τ of $\mathscr{F}$ is a t-set (t for true) when it is closed under modus ponens, i.e. when for each $a,b \in \mathscr{F}$ the following is satisfied: If aRb and $a \in \tau$, then $b \in \tau$. The class of all t-sets of $\langle \mathscr{F}, R\rangle$ will be denoted by $\mathfrak{T}(R)$, and it will be called the set of valuations compatible with R. Analogously, a fuzzy subset f is a f-set (f for false) when it is closed under modus tollens, i.e. for each $a, b \in \mathscr{F}$ the following is satisfied: If aRb and $b \in f$, the $a \in f$. The class of all f-sets of $\langle \mathscr{F}, R\rangle$ will be denoted by $\mathfrak{F}(R)$.*
Having fixed a t-norm $*$, the valuations which are compatible with an implication relation can be extended to the multiple-valued case:

DEFINITION 5. [6] *Given a fuzzy relation R on $\mathscr{F}$, a fuzzy subset τ of $\mathscr{F}$ is a t^*-set (t for true) when it is closed under modus ponens, i.e. when for each $a, b \in \mathscr{F}$, $\tau(a)^* R(a, b) \leq \tau(b)$ is satisfied. The class of all t^*-sets of $\langle \mathscr{F}, R\rangle$ will be denoted by $\mathfrak{T}^*(R)$, and it will be called the set of valuations $*$-compatible with R. Analogously, a fuzzy subset f is a f^*-set (f for false) when it is closed under modus tollens, i.e. for each $a, b \in \mathscr{F}$, $f(b)^* R(a, b) \leq f(a)$ is satisfied. The class of all f^*-sets of $\langle \mathscr{F}, R\rangle$ will be denoted by $\mathfrak{F}^*(R)$.*

Remark 2. [12] If R is a binary relation, the $\mathfrak{T}(R) = \{s \in \mathfrak{T}^*(R)/s$ is a non-fuzzy subset$\}$, and $\mathfrak{F}(R) = \{s \in \mathfrak{F}^*(R)/s$ is a non-fuzzy subset$\}$ for any t-norm $*$.

The general properties of $\mathfrak{T}^*(R)$, $\mathfrak{F}^*(R)$, $\mathfrak{T}(R)$ and $\mathfrak{F}(R)$ are summarized as follows.

PROPOSITION 2.

1. *$\mathfrak{T}^*(R) = \mathfrak{F}^*(R_d)$ and $\mathfrak{F}^*(R) = \mathfrak{T}^*(R_d)$, where R_d denotes the dual relation $R_d(a, b) = R(b, a)$.*
2. *$\mathfrak{T}(R) = \mathfrak{F}(R_d)$ and $\mathfrak{F}(R) = \mathfrak{T}(R_d)$ where $aR_db \Leftrightarrow bRa$.*
3. *$\mathfrak{F}(R) = \{\mathscr{F} - \tau : \tau \in \mathfrak{T}(R)\}$ and $\mathfrak{T}(R) = \{\mathscr{F} - f : f \in \mathfrak{F}(R)\}$.*
4. *$\emptyset, \mathscr{F} \in \mathfrak{T}^*(R)$ and $\emptyset, \mathscr{F} \in \mathfrak{F}^*(R)$. In general, if s is constant ($s(a) = s(b)$ for every a, b in $\mathscr{F}$), then $s \in \mathfrak{T}^*(R)$ and $s \in \mathfrak{F}^*(R)$.*
5. *$\emptyset, \mathscr{F} \in \mathfrak{T}(R)$ and $\emptyset, \mathscr{F} \in \mathfrak{F}(R)$.*
6. *If $*$ is lower semicontinuous, $\mathfrak{T}^*(R)$ and $\mathfrak{F}^*(R)$ are closed under arbitrary fuzzy unions and fuzzy intersections.*
7. *$\mathfrak{T}(R)$ and $\mathfrak{F}(R)$ are closed under arbitrary fuzzy unions and fuzzy intersections.*
8. *If $R_1 \subseteq R_2$ then $\mathfrak{T}^*(R_2) \subseteq \mathfrak{T}^*(R_1)$.*
9. *If $R_1 \subseteq R_2$ then $\mathfrak{T}(R_2) \subseteq \mathfrak{T}(R_1)$.*

Proof

1. and 2. see [7].

3. and 4. are obvious.

5. is a consequence of 4.

6. Let $A \subseteq \mathfrak{T}^*(I)$ be and set $\tau_0 = \bigcap A$ and $\tau_1 = \bigcup A$. Then $\tau_0(a) = \inf\{\tau(a): a \in A\}$ and $\tau_1(a) = \sup\{\tau(a): \tau \in A\}$, $\forall a \in \mathcal{F}$. Therefore, for each $\tau \in A, \tau_0(a)^* I(a,b) \leq \tau(a)^* I(a,b) \leq \tau(b)$, $\forall a, b \in \mathcal{F}$. Hence $\tau_0(a)^* I(a,b) \leq \inf\{\tau(b): \tau \in A\} = \tau_0(b)$ $\forall a, b \in \mathcal{F}$ and $\tau_0 \in \mathfrak{T}^*(I)$. On the other hand, from the lower semicontinuity of $\tau_1(a)^* I(a,b) = \sup\{\tau(a): \tau \in A\}^* I(a,b) = \sup\{\tau(a)^* I(a,b): \tau \in A\} \leq \sup\{\tau(b): \tau \in A\} = \tau_1(b)$, $\forall a, b \in \mathcal{F}$ immediately follows and so, $\tau_1 \in \mathfrak{T}^*(I)$, and the proof is complete.

7. is a consequence of 6.

8. Let $R_1 \subseteq R_2$ and $\tau \in \mathfrak{T}^*(R_2)$. Then $\tau(a)^* R_1(a,b) \leq \tau(b)\tau(a)^* R_2(a,b) \leq \tau(b)$ and $\tau \in \mathfrak{T}^*(R_1)$.

9. is a consequence of 8. ■

We can express the structure of $\mathfrak{T}(R)$ ($\mathfrak{T}^*(R)$) by means of the concept of (fuzzy) closure and co-closure system:

DEFINITION 6. [8] *A family $\mathcal{C}$ of (fuzzy) subsets of $\mathcal{F}$ is said to be a (fuzzy) closure system on $\mathcal{F}$ if $\mathcal{F} \in \mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary (fuzzy) intersections, i.e. $\bigcap A$ is in $\mathcal{C}$ if $A \subseteq \mathcal{C}$. Dually, $\mathcal{C}$ is said to be a (fuzzy) co-closure system on $\mathcal{F}$ if $\varnothing \in \mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary (fuzzy) unions ($\bigcup A$ is in $\mathcal{C}$ if $A \subseteq \mathcal{C}$). Both the (fuzzy) closure and (fuzzy) co-closure systems will be called (fuzzy) CC-systems in short.*

Remark 3. [12] If $\mathcal{C}$ is a fuzzy closure (co-closure) system on $\mathcal{F}$, then for each $\alpha \in [0,1]$ the family $\mathcal{C}_\alpha$ defined by $\mathcal{C}_\alpha = [s_\alpha : s \in \mathcal{C}\}$, s_α denotes the α-cut of s, is a "crisp" closure (co-closure) system.

THEOREM 1.

1. *$\mathfrak{T}(R)$ and $\mathfrak{F}(R)$ are two CC-systems on $\mathcal{F}$.*
2. *If $*$ is a lower semicontinuous t-norm, $\mathfrak{T}^*(R)$ and $\mathfrak{F}^*(R)$ are two fuzzy CC-systems on $\mathcal{F}$.*

2.3. Relation Between Preorders and CC-Systems

Now, we can prove that in the bivalued case both processes, induction of implication relations and setting of possible valuations, are self-inverse. However, in the multiple-valued case this does not hold, since they are self-inverse in only one way.

PROPOSITION 3.

1. $\mathcal{C} \subseteq \mathfrak{T}^*(I^*(\mathcal{C}))$.
2. $R \subseteq I^*(\mathfrak{T}^*(R))$.

Proof:

1. If $s \in \mathscr{C}$ then $s(a) \otimes s(b) \geq I^*(\mathscr{C})(a, b)$, hence $s(b) \geq s(a)I^*(\mathscr{C})(a, b)$, and $s \in \mathfrak{T}^*(I^*(\mathscr{C}))$.
2. If $s \in \mathfrak{T}^*(R)$ then $s(b) \geq s(a)^*R(a, b)$, hence $R(a, b) \leq s(a) \otimes s(b)$ and $R(a, b) \leq \inf\{s(a) \otimes s(b)/s \in \mathfrak{T}^*(R)\} = I^*(\mathfrak{T}^*(R))(a, b)$. ∎

COROLLARY 1.

1. $\mathscr{C} \subseteq \mathfrak{T}(I(\mathscr{C}))$.
2. $R \subseteq I(\mathfrak{T}(R))$.

PROPOSITION 4. *If R is a $*$-fuzzy preorder on $\mathscr{F}$, then $I^*(\mathfrak{T}^*(R)) = R$.*

Proof: If $s \in \mathfrak{T}^*(R)$, for each $a, b \in \mathscr{F}$,

$$s(a)^* R(a, b) \leq s(b),$$

hence $R(a, b) \leq s(a) \oplus s(b)$. Therefore

$$R(a, b) \leq \operatorname*{Inf}_{s \in \mathfrak{T}^*(R)} s(a) \oplus s(b) = I^*(\mathfrak{T}^*(R))(a, b).$$

On the other hand, from $R(a, x)^*R(x, y) \leq R(a, y), s_a(x) = R(a, x) \in \mathfrak{T}^*(R)$ follows. Consequently,

$$I^*(\mathfrak{T}^*(R))(a, b) \leq s_a(a) \oplus s_a(b) = 1 \oplus s_a(b) = R(a, b). \quad ∎$$

Remark 4. This proposition can be seen as an extension of the representation theorem of fuzzy preorders [4]:

REPRESENTATION THEOREM OF FUZZY PREORDERS. *Let $*$ be a lower semicontinuous t-norm. If $\mathscr{C}$ is a family of fuzzy subsets of $\mathscr{F}$, then $I^*(\mathscr{C})(a, b) = \inf\{s(a) \otimes s(b): s \in \mathscr{C}\}$ is a $*$-fuzzy preorder on $\mathscr{F}$. Conversely, if I is a $*$-fuzzy preorder on $\mathscr{F}$ then there exists a family of fuzzy subsets of $\mathscr{F}$, $\mathscr{C}$, such that $I^*(\mathscr{C}) = I$.*

In proposition 4 it has been established that if R is a fuzzy preorder, then there exists a family $\mathscr{C} = \mathfrak{T}^*(R)$ of fuzzy subsets such that $I^*(\mathscr{C}) = R$.

COROLLARY 2. *If R is a preorder, then $I(\mathfrak{T}(R)) = R$.*

REPRESENTATION THEOREM OF PREORDERS. *If $\mathscr{C}$ is a family of subsets of a set $\mathscr{F}$, the $I(\mathscr{C})$ is a preorder on $\mathscr{F}$. Conversely, if I is a preorder on $\mathscr{F}$ then there exists a family of $\mathscr{C}$ of subsets of $\mathscr{F}$ such that $I(\mathscr{C}) = I$.*

In corollary 2 it has been established that if R is a preorder, then there exists a family $\mathscr{C} = \mathfrak{T}(R)$ of subsets such that $I(\mathscr{C}) = R$.

PROPOSITION 5. *If $\mathscr{C}$ is a CC-system on $\mathscr{F}$ then $\mathfrak{T}(I(\mathscr{C})) = \mathscr{C}$.*

Proof: Let $s \in \mathscr{C}$. If $aI(\mathscr{C})b$ then $b \in s'$ for each $s' \in \mathscr{C}$ such that $a \in s'$, hence if $a \in s$ then $b \in s$ and $s \in \mathfrak{T}(I(\mathscr{C}))$. Conversely, let $\tau \in \mathfrak{T}(I(\mathscr{C}))$. For each $a \in \mathscr{F}$, set

$$\mathscr{C}_a = \{s \in \mathscr{C} : a \in s\},$$

is not empty since $\mathscr{F} \in \mathscr{C}_a$. Let $s_a = \bigcap \mathscr{C}_a \in \mathscr{C}$. If $b \in s_a$, then $b \in s$ for every $s \in \mathscr{C}$ such that $a \in s$, hence $aI(\mathscr{C})b$ and $b \in \tau$; consequently $s_a \subseteq \tau$ for each $a \in \tau$. Let $s = \bigcup_{a \in \tau} s_a \in \mathscr{C}$, then $s \subseteq \tau$. Let $b \in \tau$, then $b \in s_b$ and $b \in s$, hence $\tau \subseteq s$ and $\tau = s \in \mathscr{C}$. ■

PROPOSITION 6. *If $\mathscr{C}$ is a fuzzy CC-system, it is not always $\mathscr{C} = \mathfrak{T}^*(I(\mathscr{C}))$.*

Proof: Let us consider the case $\mathscr{F} = \{a, b, c\}$ and

$$\mathscr{C} = \{\{a/\alpha, b/\beta, c/\gamma\} : \alpha = \beta = \gamma \quad \text{or} \quad \alpha = 0, \beta = 1/2, \gamma = 1\}.$$

With the t-norm Π (product):

$$\begin{array}{ll} I^{\pi}(\mathscr{C})(a, b) = 1 & I^{\pi}(\mathscr{C})(b, a) = 0 \\ I^{\pi}(\mathscr{C})(a, c) = 1 & I^{\pi}(\mathscr{C})(c, a) = 0 \\ I^{\pi}(\mathscr{C})(c, b) = 1/2 & I^{\pi}(\mathscr{C})(b, c) = 1 \end{array}$$

hence τ is a t-π-set iff

$$\tau(b) \geq \tau(a), \quad \tau(c) \geq \tau(a) \quad \text{and} \quad \tau(b) \geq \tau(c)/2.$$

Then, if $\tau(a) = 1/4$, $\tau(b) = 3/4$ and $\tau(c) = 1/4$, $\tau \in \mathfrak{T}^{\pi}(I^{\pi}(\mathscr{C}))$, but τ does not belong to $\mathscr{C}^c$ since $\tau(c) < \tau(b)$. ■

Remark 5. [9] The family of all CC-systems on $\mathscr{F}$ is a complete lattice under set inclusion where

$$\operatorname*{Inf}_i \mathscr{C}_i = \bigcap_i \mathscr{C}_i = \{s \subseteq \mathscr{F} : s \in \mathscr{C}_i \text{ for all } i\},$$

$$\sup_i \mathscr{C}_i = \operatorname{Inf}\{\mathscr{C} : \mathscr{C}_i \subseteq \mathscr{C}, \text{for all } i\}.$$

PROPOSITION 7. *The family of all fuzzy CC-systems on $\mathscr{F}$ is a complete lattice under set inclusion where*

$$\operatorname*{Inf}_i \{\mathscr{C}_i\} = \bigcap_i \mathscr{C}_i = \{s : \mathscr{F} \to [0,1] : s \in \mathscr{C}_i \textit{ for all } i\},$$

$$\sup\{\mathscr{C}_i\} = \operatorname{Inf}\{\mathscr{C} : \mathscr{C}_i \subseteq \textit{ for all } i\}.$$

Proof: It is enough to prove that for any family $\{\mathscr{C}_i\}$ of fuzzy CC-systems, its intersection $\bigcap_i \mathscr{C}_i$ is a fuzzy CC-system too.
For any i, $\mathscr{F}, \emptyset \in \mathscr{C}_i$ and then $\mathscr{F}, \emptyset \in \bigcap_i \mathscr{C}_i$. Now let $A \subseteq \bigcap_i \mathscr{C}_i$ be. The $A \subseteq \mathscr{C}_i$ for each i, hence both $\bigcap A$, $\bigcup A$ belong to $\mathscr{C}_i$ for each i as $\mathscr{C}_i$ is a fuzzy CC-system. Thus both, $\bigcap A$, $\bigcup A$ are in $\bigcap_i \mathscr{C}_i$ too. ∎

Remark 7. [9] The family of all preorders on $\mathscr{F}$ is a complete lattice under the ordering

$$R_1 \leq R_2 \qquad \text{iff } R_1 \subseteq R_2,$$

where

$$\operatorname*{Inf}_i R_i = \bigcap_i R_i, \ \sup_i R_i = \operatorname{Inf}\{R: R_i \subseteq R, \text{for all } i\}.$$

PROPOSITION 8. *Given a continuous t-norm* $*$, *the family of all* $*$*-fuzzy preorders on* $\mathscr{F}$ *is a complete lattice under the ordering*

$$I_1 \leq I_2 \qquad \textit{iff} \quad I_1(a,b) \leq I_2(a,b) \textit{ for each } a,b \in \mathscr{F}.$$

where

$$\operatorname*{Inf}_i I_i = \bigcap_i I_i;$$

$$\sup_i \mathscr{C}_i = \operatorname{Inf}\{\mathscr{C}: \mathscr{C}_i \subseteq \mathscr{C}, \textit{for all } i\}.$$

Proof: It is enough to prove that if I_i is a $*$-fuzzy preorder for each i, then $\bigcap_i I_i$ is also a $*$-fuzzy preorder. If $a, b, c \in \mathscr{F}$, for each j

$$I_j(a,b)^* I_j(b,c) \leq I_j(a,c)$$

is verified, hence for each j

$$\cap I_i(a,b)^* \cap I_i(b,c) \leq I_j(a,c)$$

and

$$\bigcap_i I_i(a,b)^* \bigcap_i I_i(b,c) \leq \bigcap_i I_i(a,c). \qquad ∎$$

THEOREM 2. *The complete lattice of all CC-systems on* $\mathscr{F}$ *is in one to one and onto correspondence with the complete lattice of all preorders on* $\mathscr{F}$ *by the maps*

$$R \to \mathfrak{T}(R) \quad \textit{and} \quad \mathscr{C} \to I(\mathscr{C}).$$

THEOREM 3. *Given a continuous t-norm* $*$, *the complete lattice of all* $*$*-fuzzy preorders on* $\mathscr{F}$ *is one to one correspondence with complete lattice of* *all fuzzy CC-systems on $\mathcal{F}$ by the maps*

$$R \to \mathfrak{T}^*(R) \quad and \quad \mathscr{C} \to I^*(\mathscr{C}).$$

2.4. When Do Two Different Families of Possible Valuations Induce the Same Implication Relation?

Two questions are now considered: (1) When do two different families of possible valuations produce the same material implication?, i.e., when does $I(\mathscr{C}_1) = I(\mathscr{C}_2)$?, and (2) When do two different partial implication relations produce the same possible valuations?, i.e., when does $\mathfrak{T}(R_1) = \mathfrak{T}(R_2)$?.

Remark [8] Given an arbitrary family $\mathscr{C}$ of (fuzzy) subsets of $\mathcal{F}$, the infimum of all (fuzzy) CC-systems that contain $\mathscr{C}$ will be named the smallest (fuzzy) CC-systems which contains $\mathscr{C}$. It will be denoted $\mathscr{C}^c$.

Remark [9] Given a (fuzzy) relation R on $\mathcal{F}$, the infimum of all (*-fuzzy) preorders on $\mathcal{F}$ that contains R is the smallest (*-fuzzy) preorder on $\mathcal{F}$ which contains R. It will be denoted $R^c(R^c_*)$.

PROPOSITION 9.

1. $\mathfrak{T}(I(\mathscr{C})) = \mathscr{C}^c$.
2. $I(\mathfrak{T}(R)) = R^c$.
3. $I^*(\mathfrak{T}^*(R)) = R^c_*$.

Proof:

1. From proposition 3 $\mathfrak{T}(I(\mathscr{C}))$ is a CC-system that contains $\mathscr{C}$, hence $\mathscr{C}^c \subseteq \mathfrak{T}(I(\mathscr{C}))$. If $\mathscr{C}_1$ is a CC-system that contains $\mathscr{C}$, then $I(\mathscr{C}_1) \subseteq I(\mathscr{C})$ and $\mathfrak{T}(I(\mathscr{C})) \subseteq \mathfrak{T}(I(\mathscr{C}_1)) = \mathscr{C}_1$, hence $\mathfrak{T}(I(\mathscr{C}))$ is included in every CC-system that contains $\mathscr{C}$ and $\mathfrak{T}(I(\mathscr{C})) = \mathscr{C}^c$.
2. From proposition 3 $I(\mathfrak{T}(R))$ is a preorder that contains R, hence $R^c \subseteq I(\mathfrak{T}(R))$. If R_1 is a preorder that contains R, then $\mathfrak{T}(R_1) \subseteq \mathfrak{T}(R)$ and $I(\mathfrak{T}(R)) \subseteq I(\mathfrak{T}(R_1)) = R_1$, hence $I(\mathfrak{T}(R))$ is included in every preorder that contains R and $I(\mathfrak{T}(R)) = R^c$.
3. The proof is analogous to (2). ■

PROPOSITION 10.

1. $I(\mathscr{C}) = I(\mathscr{C}^c)$
2. $\mathfrak{T}(R) = \mathfrak{T}(R^c)$.
3. $\mathfrak{T}^*(R) = \mathfrak{T}^*(R^c_*)$.

Proof:

1. From $\mathscr{C}^c = \mathfrak{T}(I(\mathscr{C}))$, $I(\mathscr{C}^c) = I(\mathfrak{T}(I(\mathscr{C}))) = I(\mathscr{C})$.
2. From $R^c = I(\mathfrak{T}(R))$, $\mathfrak{T}(R^c) = \mathfrak{T}(I(\mathfrak{T}(R))) = \mathfrak{T}(R)$.
3. The proof is analogous to (2). ■

THEOREM 4.

1. *Given two families $\mathscr{C}_1, \mathscr{C}_2$ of subsets of $\mathscr{T}$,*
$$I(\mathscr{C}_1) = I(\mathscr{C}_2) \qquad \textit{iff } \mathscr{C}_1^c = \mathscr{C}_1^c.$$
2. *Given two binary relations R_1, R_2 on $\mathscr{T}$,*
$$\mathfrak{T}(R_1) = \mathfrak{T}(R_2) \qquad \textit{iff } R_1^c = R_2^c.$$
3. *Given two fuzzy relation R_1, R_2 on $\mathscr{T}$,*
$$\mathfrak{T}^*(R_1) = \mathfrak{T}^*(R_2) \qquad \textit{iff } R_{1*}^c = R_{2*}^c.$$

Proof:

1. If $I(\mathscr{C}_1) = I(\mathscr{C}_2)$ then $\mathscr{C}_1^c = \mathfrak{T}(I(\mathscr{C}_1)) = \mathfrak{T}(I(\mathscr{C}_2^c)) = \mathscr{C}_2^c$. Conversely, if $\mathscr{C}_1^c = \mathscr{C}_2^c$ then $I(\mathscr{C}_1) = I(\mathscr{C}_1^c) = I(\mathscr{C}_2^c) = I(\mathscr{C}_2)$.
2. If $\mathfrak{T}(R_1) = \mathfrak{T}(R_2)$ then $R_1^c = I(\mathfrak{T}(R_1)) = I(\mathfrak{T}(R_2^c)) = R_2^c$. Conversely, if $R_1^c = R_2^c$ then $\mathfrak{T}(R_1) = \mathfrak{T}(R_1^c) = \mathfrak{T}(R_2^c) = \mathfrak{T}(R_2)$.
3. If $\mathfrak{T}^*(R_1) = \mathfrak{T}^*(R_2)$ then $R_{1*}^c = I^*(\mathfrak{T}^*(R_{1*}^c)) = I^*(\mathfrak{T}^*(R_{2*}^c)) = R_{2*}^c$. Conversely, if $R_{1*}^c = R_{2*}^c$ then $\mathfrak{T}(R_1) = \mathfrak{T}(R_{1*}^c) = \mathfrak{T}(R_{2*}^c) = \mathfrak{T}(R_2)$. ■

Remark 10 It is not possible to assure that $I^*(\mathscr{C}_1) = I^*(\mathscr{C}_2)$ iff $\mathscr{C}_1^c = \mathscr{C}_2^c$. So, the problem: "When is $I^*(\mathscr{C}_1) = I^*(\mathscr{C}_2)$?" is still open.

EXAMPLE 1. Let us consider the case $\mathscr{T} = \{a, b\}$ and
$$t_1 = \{a, b\} \quad t_2 = \{a\} \quad t_3 = \{b\} \quad t_4 = \varnothing$$

$\mathscr{C}$	$I(\mathscr{C})$	$\mathscr{C}^c$
$\{t_1\}$	$a \leftrightarrow b$	$\{t_1, t_4\}$
$\{t_2\}$	$a \leftarrow b$	$\{t_1, t_2, t_4\}$
$\{t_3\}$	$a \rightarrow b$	$\{t_1, t_3, t_4\}$
$\{t_4\}$	$a \leftrightarrow b$	$\{t_1, t_4\}$
$\{t_1, t_2\}$	$a \leftarrow b$	$\{t_1, t_2, t_4\}$
$\{t_1, t_3\}$	$a \rightarrow b$	$\{t_1, t_3, t_4\}$
$\{t_1, t_4\}$	$a \leftrightarrow b$	$\{t_1, t_4\}$
$\{t_2, t_3\}$	$a\ b$	$\{t_1, t_2, t_3, t_4\}$
$\{t_2, t_4\}$	$a \leftarrow b$	$\{t_1, t_2, t_4\}$
$\{t_3, t_4\}$	$a \rightarrow b$	$\{t_1, t_3, t_4\}$
$\{t_2, t_3, t_4\}$	$a\ b$	$\{t_1, t_2, t_3, t_4\}$
$\{t_1, t_3, t_4\}$	$a \rightarrow b$	$\{t_1, t_3, t_4\}$
$\{t_1, t_2, t_4\}$	$a \leftarrow b$	$\{t_1, t_2, t_4\}$
$\{t_1, t_2, t_3\}$	$a\ b$	$\{t_1, t_2, t_3, t_4\}$
$\{t_1, t_2, t_3, t_4\}$	$a\ b$	$\{t_1, t_2, t_3, t_4\}$

3. PROBABILISTIC INDUCTION—EXTENSION OF A PREORDER

In the previous section an induction procedure was considered in order to obtain an implication relation from the family $\mathscr{C}$ of possible valuations on $\mathscr{F}$. Now, let us consider the case in which each possible valuation has probabilistic information associated to it:

$$\mathscr{C} : \mathbb{P}(\mathscr{F}) \to [0,1] \qquad \sum_{s \subseteq \mathscr{F}} \mathscr{C}(s) = 1. \tag{1}$$

where $\mathscr{C}(s)$ represents the probability of the sentence "s is the set of true propositions".

A probability $\text{Prob}_{\mathscr{C}}$ on the free Boolean algebra $\mathscr{B}(\mathscr{F})$ [10] generated by $\mathscr{F}$ can be induced from $\mathscr{C}$ in two steps:

1. For every $s \in \mathscr{F}$ and $\alpha \in \mathscr{B}(\mathscr{F})$ the relation $sV\alpha$ (say s verifies α) is defined recursively by (see [11, 14]):
   - **(a)** If $\alpha \in \mathscr{F}$, then $sV\alpha$ iff $\alpha \in s$,
   - **(b)** If $\alpha = 0$, then never $sV\alpha$,
   - **(c)** If $\alpha = 1$, then always $sV\alpha$,
   - **(d)** If $\alpha = \neg \beta$, then $sV\alpha$ iff not $sV\beta$,
   - **(e)** If $\alpha = \beta \wedge \gamma$, then $sV\alpha$ iff $sV\beta$ and $sV\gamma$,
   - **(f)** If $\alpha = \beta \vee \gamma$, then $sV\alpha$ iff $sV\beta$ or $sV\gamma$.
2. Now the application $\text{Prob}_{\mathscr{C}}$: $\mathscr{B}(\mathscr{F}) \to [0,1]$, is constructed according to

$$\text{Prob}_{\mathscr{C}}(\alpha) = \sum_{sV\alpha} \mathscr{C}(s)$$

From (a) and (1) $\text{Prob}_{\mathscr{C}}(\neg \alpha) = 1 - \text{Prob}_{\mathscr{C}}(\alpha)$ follows. From (b), (e), and (f) it is easy to show that $\alpha \wedge \beta = 0$ implies $\text{Prob}_{\mathscr{C}}(\alpha \vee \beta) = \text{Prob}_{\mathscr{C}}(\alpha) + \text{Prob}_{\mathscr{C}}(\beta)$. Thus $\text{Prob}_{\mathscr{C}}$ is a probability on $\mathscr{B}(\mathscr{F})$.

Now the implication is taken by:

$$I_{\text{prob}}(\mathscr{C})(a,b) =^{def} \text{Prob}_{\mathscr{C}}(b/a),$$

The interpretation is: the extent to which b can be implied by a is the conditional probability of b relative to a.

EXAMPLE 2. Let $\mathscr{F} = \{a, b, c\}$ and suppose that a statistical study has given the following results:

- In 80% of cases a and b are true and c is false,
- In 15% of cases a is true and b and c are false,
- In 5% of cases b and c are true and a is false,

then $\mathscr{C}$ is defined as:

$$\mathscr{C}(\{b\}) = \mathscr{C}(\{c\}) = \mathscr{C}(\{a,c\}) = \mathscr{C}(\emptyset) = \mathscr{C}(\{a,b,c\}) = 0;$$
$$\mathscr{C}(\{a,b\}) = 0.8; \quad \mathscr{C}(\{a\}) = 0.15; \quad \mathscr{C}(\{b,c\}) = 0.05,$$

which produces the following probability over the free Boolean algebra generated by $\mathscr{F}$:

$$\text{Prob}(a) = 0.95; \quad \text{Prob}(b) = 0.85; \quad \text{Prob}(c) = 0.05;$$
$$\text{Prob}(a \wedge b) = 0.8; \quad \text{Prob}(a \wedge c) = 0;$$
$$\text{Prob}(b \wedge c) = 0.05; \quad \text{Prob}(a \wedge b \wedge c) = 0.$$

and then the conditional probabilities:

$$\begin{array}{ll} \text{Prob}(b/a) = 0.84; & \text{Prob}(c/a) = 0; \\ \text{Prob}(a/b) = 0.94; & \text{Prob}(c/b) = 0.05; \\ \text{Prob}(a/c) = 0; & \text{Prob}(b/c) = 1. \end{array}$$
$$\text{Prob}(b/a \wedge c) = \text{Prob}(a/b \wedge c) = \text{Prob}(c/b \wedge c) = 0.$$

THEOREM 5. *Given the probabilistic information on* $\mathscr{F}$

$$\mathscr{C}: \mathbb{P}(\mathscr{F}) \to [0,1],$$

the preorder (implication relation) $I(\mathscr{C}_0)$ *associated to* $\mathscr{C}_0 = \{s: \mathscr{C}(s) > 0\}$ *verifies*

$$aI(\mathscr{C}_0)b \qquad \textit{iff } Prob_{\mathscr{C}}(b/a) = 1.$$

Proof: According to its definition $\text{Prob}_{\mathscr{C}}(b/a) = 1$ iff $\text{Prob}_{\mathscr{C}}(a \wedge b) = \text{Prob}_{\mathscr{C}}(a)$. On its turn $\text{Prob}_{\mathscr{C}}(a \wedge b) = \text{Prob}_{\mathscr{C}}(a)$ iff $\Sigma_{sVa \wedge b}\mathscr{C}(s) = \Sigma_{sVa}\mathscr{C}(s)$ and this last equality holds iff for each $s \in \mathscr{C}_0$ such that $a \in s$, also $b \in s$. In conclusion, $\text{Prob}_{\mathscr{C}}(b/a) = 1$ iff $aI(\mathscr{C}_0)b$. ∎

Remark 11. The justification of taking $\mathscr{C}_0$ is that it represents the information forgetting the probability: A subset $s \in \mathscr{C}_0$ if and only if s has positive probability, if and only if in a case s is the set of true facts.

COROLLARY 3. $I(\mathscr{C}_0)$ *is the* 1*-cut of* $I_{prob}(\mathscr{C})$.

DEFINITION 7. *Given a binary relation R on a finite set* $\mathscr{F}$, *let us consider the mapping* $\mathscr{C}$: $\mathbb{P}(\mathscr{F}) \to [0,1]$, *given by*

$$\mathscr{C}(s) = \begin{cases} 1/\text{Card}(\mathfrak{T}(R)) & \textit{if } s \in \mathfrak{T}(R) \\ 0 & \textit{otherwise}. \end{cases}$$

The probability associated to $\mathscr{C}$ *will be called the probabilistic extension of R and will be denoted* $Prob_R$.

EXAMPLE 3. Let $\mathcal{F} = \{a, b, c\}$ and the relations defined by:

R_1	R_2	R_3
$a \to b$	$a \to b$	$a \to c$
$b \to c$	$a \to c$	

then

$$\mathfrak{T}(R_1) = \{\{a,b,c\},\{b,c\},\{c\},\varnothing\}$$

$$\mathfrak{T}(R_2) = \{\{a,b,c\},\{b\},\{b,c\},\{c\},\varnothing\}$$

$$\mathfrak{T}(R_3) = \{\{a,c\},\{a,b,c\},\{b\},\{c\},\{b,c\},\varnothing\}$$

and the respective associated probabilities are:

$\mathrm{Prob}_{R1}(a) = 1/4$	$\mathrm{Prob}_{R1}(b) = 1/2$	$\mathrm{Prob}_{R1}(c) = 3/4$
$\mathrm{Prob}_{R2}(a) = 1/5$	$\mathrm{Prob}_{R2}(b) = 3/5$	$\mathrm{Prob}_{R_2}(c) = 3/5$
$\mathrm{Prob}_{R3}(a) = 1/3$	$\mathrm{Prob}_{R3}(b) = 1/3$	$\mathrm{Prob}_{R3}(c) = 2/3$

producing the probabilistic extensions:

$\mathrm{Prob}_{R1}(b/a) = 1$	$\mathrm{Prob}_{R2}(b/a) = 1$	$\mathrm{Prob}_{R3}(b/a) = 1/2$
$\mathrm{Prob}_{R1}(c/a) = 1$	$\mathrm{Prob}_{R2}(c/a) = 1$	$\mathrm{Prob}_{R3}(c/a) = 1$
$\mathrm{Prob}_{R1}(a/b) = 1/2$	$\mathrm{Prob}_{R2}(a/b) = 1/3$	$\mathrm{Prob}_{R3}(a/b) = 1/3$
$\mathrm{Prob}_{R1}(c/b) = 1$	$\mathrm{Prob}_{R2}(c/b) = 2/3$	$\mathrm{Prob}_{R3}(c/b) = 2/3$
$\mathrm{Prob}_{R1}(a/c) = 1/3$	$\mathrm{Prob}_{R2}(a/c) = 1/3$	$\mathrm{Prob}_{R3}(a/c) = 2/3$
$\mathrm{Prob}_{R1}(b/c) = 2/3$	$\mathrm{Prob}_{R2}(b/c) = 2/3$	$\mathrm{Prob}_{R3}(b/c) = 1/3$

THEOREM 6. *Given a binary relation R on $\mathcal{F}$ for each $a, b \in \mathcal{F}$, $Prob_R(b/a) = 1$ iff $aR^c b$. Consequently, if R is a preorder then $Prob_R(b/a) = 1$ iff aRb.*

Proof: From $\mathscr{C}_0 = \mathfrak{T}(R)$, $I(\mathscr{C}_0) = I(\mathfrak{T}(R)) = R^c$ follows. ∎

This last result justifies the name of "probabilistic extension" of the relation that we will adopt for the conditional $\mathrm{Prob}_R(./.)$.

4. POSSIBILISTIC INDUCTION

In the previous section a probabilistic induction was developed in order to obtain a probabilistic implication from probabilistic information about the possible true values on a universe of propositions $\mathcal{F}$. Now, we will consider the case when possibilistic information, $\mathscr{C}$: $\mathbb{P}(\mathcal{F}) \to [0,1]$, is given, where $\mathbb{P}(\mathscr{F})$ stands for the family of all the subsets of $\mathscr{F}$ aned $\mathscr{C}(s)$ represents the possibility of the sentence "s is the set of true propositions".

A possibility measure $\mathrm{Poss}_{\mathscr{C}}$ and a necessity measure $\mathrm{Nec}_{\mathscr{C}}$ on the free Boolean algebra $\mathscr{B}(\mathscr{F})$ generated by $\mathscr{F}$ can be induced from $\mathscr{C}$ by using the binary relation V defined in the section 2:

$$\mathrm{Poss}_{\mathscr{C}}(\alpha) = \sup_{sV\alpha} \mathscr{C}(s), \quad \mathrm{Nec}_{\mathscr{C}}(\alpha) = 1\text{-}\mathrm{Poss}_{\mathscr{C}}(\neg\,\alpha)$$

From the properties of V (see (f)) it is obvious that $\mathrm{Poss}_{\mathscr{C}}$ is a possibility measure, and $\mathrm{Nec}_{\mathscr{C}}$ is a necessity measure as the dual one.

Now a conditional possibility (and necessity) can be defined by considering the conjunction of subsets as a conditioning:

$$\mathrm{Poss}_{\mathscr{C}}(\alpha/\beta) = \mathrm{Poss}_{\mathscr{C}}(\beta \wedge \alpha),$$

$$\mathrm{Nec}_{\mathscr{C}}(\alpha/\beta) = 1 - \mathrm{Poss}_{\mathscr{C}}(\neg\,\alpha/\beta) = 1 - \mathrm{Poss}_{\mathscr{C}}(\beta \wedge \neg\,\alpha),$$

It is obvious that for every $\alpha, \beta \in \mathscr{B}(\mathscr{F})$, $\mathrm{Poss}_{\mathscr{C}}(\alpha/\beta) = \mathrm{Poss}_{\mathscr{C}}(\beta/\alpha)$ and

$$\sup_{\alpha} \min(\mathrm{Poss}_{\mathscr{C}}(\alpha), \mathrm{Poss}_{\mathscr{C}}(\beta/\alpha)) = \mathrm{Poss}_{\mathscr{C}}(\beta)$$

Now the implication is taken by:

$$I_{\mathrm{poss}}(\mathscr{C})(a,b) =^{def} \mathrm{Nec}_{\mathscr{C}}(b/a),$$

The interpretation is: the extent to which b can be implied by a is the conditional necessity of b relative to a.

EXAMPLE 4. Let us consider $\mathscr{F} = \{a, b, c\}$ and the following possibilistic information:

$$\mathscr{C}(\varnothing) = \mathscr{C}(\{a\}) = \mathscr{C}(\{b\}) = \mathscr{C}(\{c\}) = \mathscr{C}(\{a,c\}) = \mathscr{C}(\{a,b,c\}) = 0;$$
$$\mathscr{C}(\{a,b\}) = 0.6; \quad \mathscr{C}(\{a,b,c\}) = 0.7; \quad \mathscr{C}(\{b,c\}) = 0.8.$$

The induced possibility and necessity on the free Boolean algebra generated by $\mathscr{F}$ are:

$\mathrm{Poss}(a) = 0.7$	$\mathrm{Poss}(b) = 0.8$	$\mathrm{Poss}(c) = 0.8$	
$\mathrm{Poss}(a \wedge b) = 0.7$	$\mathrm{Poss}(a \wedge c) = 0.7$	$\mathrm{Poss}(b \wedge c) = 0.8$	$\mathrm{Poss}(a \wedge b \wedge c) = 0.7$
$\mathrm{Nec}(a) = 0.2$	$\mathrm{Nec}(b) = 1$	$\mathrm{Nec}(c) = 0.4$	
$\mathrm{Nec}(a \wedge b) = 0.2$	$\mathrm{Nec}(a \wedge c) = 0.2$	$\mathrm{Nec}(b \wedge c) = 0.4$	$\mathrm{Nec}(a \wedge b \wedge c) = 0.2,$

and thus the conditional measures:

$\mathrm{Poss}(b/a) = \mathrm{Poss}(a/b) = 0.7$; $\mathrm{Poss}(c/a) = \mathrm{Poss}(a/c) = 0.7$

$\mathrm{Poss}(c/b) = \mathrm{Poss}(b/c) = 0.8$; $\mathrm{Poss}(b/a \wedge c) = \mathrm{Poss}(c/b \wedge c) =$

$\text{Poss}(a/b \wedge c) = 0.7$

$\text{Nec}(b/a) = 1$; $\text{Nec}(a/b) = 0.2$; $\text{Nec}(c/a) = 0.4$;

$\text{Nec}(a/c) = 0.2$; $\text{Nec}(b/c) = 1$; $\text{Nec}(c/b) = 0.4$.

THEOREM 7. *Given the possibilistic information on $\mathcal{F}$*

$$\mathscr{C}: \mathbb{P}(\mathcal{F}) \to [0,1],$$

the associated preorder (implication relation) $I(\mathscr{C}_0)$ verifies

$$aI(\mathscr{C})b \qquad \text{iff } Nec_{\mathscr{C}}(b/a) = 1,$$

being $\mathscr{C}_0 = \{s: \mathscr{C}(s) > 0\}$.

Proof: $\text{Nec}_{\mathscr{C}}(b/a) = 1$ iff $\text{Poss}_{\mathscr{C}}(\neg b \wedge a) = 0$. According to the definition of the conditional measures, this last equality is verified iff for each $s \in \mathscr{C}$ such that sVa, s is not related with $\neg b$ by V, but this holds iff for each $s \in \mathscr{C}$ such that $a \in s$, also $b \in s$, i.e., $aI(\mathscr{C})b$. ■

Remark 12. The justification of taking $\mathscr{C}_0$ is that it represents the information forgetting the possibility: a subset $s \in \mathscr{C}_0$ if and only if s has positive possibility, if and only if in some case s is the set of true facts.

COROLLARY 4. *$I(\mathscr{C}_0)$ is the 1-cut of $I_{poss}(\mathscr{C})$.*

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